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A technique of extracting useful informations in the computations of higher order autocorrelation functions

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Abstract : Expressions are provided for extracting higher order effects in the computation of higher order autocorrelation functions (equivalent to cumulants rather than moments) obtained by subtracting the contributions of lower order autocorrelation functions. The expressions will be useful for experimentalists wishing to get useful temporal informations from digitized time series of experimental signals.

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1. Introduction

In recording apparent noisy experimental signals, experimentalists are often interested in computing, among other quantities, autocorrelation functions to get useful information, if any, from the signal and use it to distinguish a periodic signal from a nonperiodic singal or even a chaotic (deterministic) signal from random noise. As is well known, autocorrelation functions measure the average temporal relations of the fluctuations in the signal (Born and Wolf 1979, Glauber 1963, 1965). It has characteristic shapes and properties, depending on the nature of the fluctuations. In experimental optical physics, it is a common technique to use autocorrelation functions to find optical bandwidth, short term memory features and other temporal behaviours in the optical signal (Abraham and Smith 1981, Abraham et *al* 1987).

As is well known, an autocorrelation function of order $(i+j)$ is defined as,

$$c^{ij}(\tau)=\langle I(t)^{i}I(t+\tau)^{j}\rangle \tag{1}$$

where $I(t)$ represents the strength of the signal at time t and τ represents the time delay. In many circumstances computations or measurements of higher order autocorrelation functions become essential or very important in addition to the second

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order autocorrelation function, which is commonly computed or measured in most experimental situations. As the third and higher order autocorrelation functions are more sensitive to larger pulses, they can reveal features that may be relatively hidden in the second order autocorrelation functions. In performing such computations or measurements of higher order autocorrelation functions, one wants to reveal higher order effects of that particular order that may be happening or occurring in the signal with a view to getting additional pieces of information needed to understand the complete dynamics of the system giving rise to the signal.

2. Cumulant-like autocorrelation function

It is known that, in order to get the statistics of a signal one can compute (or measure) moments, central moments or cumulants (Chuang *et al* 1980, Das 1988, Das *et al* 1989, Das and Abraham 1991). But one must compute or measure higher order cumulants (that is not moments, central moments etc.) to reveal "true" higher order effects in the statistics. This is because it is known that n-th order cumulant has been defined in such a way that it represents true characteristic n-th order moment of the distribution by subtracting the contribution of all of its lower order moments (Abraham *et al* 1987, Chuang *et al* 1980, Das 1988, Das *et al* 1989). However, in the case of higher order autocorrelation functions, as seen from the definition in eq. (1), it is clear that one cannot expect the true higher order effect of only that particular order to be present in the autocorrelation function of that order, sin e no attempt has been made in the definition to wash out the effects of its lower order autocorrelation functions.

Such a procedure of subtraction was not done for autocorrelation functions. However, as is explained below, it can be easily done if one reviews the connections among moments, cumulants and autocorrelation functions.

In defining a cumulant of n-th order as a true moment of that order, as the contribution of all lower (n-1) order moments has been subtracted to their appropriate power in a logical manner, the relations between cumulants and moments has become as provided below (given up to the 5th order) (Das 1988, Kendall and Stuart 1977)

$$K_1 = M_1, \tag{2}$$

$$K_2 = M_2 - M_1^2, \tag{3}$$

$$K_3 = M_3 - 3M_2M_1 + 2M_1^3, \tag{4}$$

$$K_4 = M_4 - 4M_3M_1 - 3M_2^2 + 12M_2M_1^2 - 6M_1^4, \tag{5}$$

$$K_5 = M_5 - 5M_4M_1 - 10M_3M_2 + 20M_3M_1^2 + 30M_2^2M_1 - 60M_2M_1^3 + 24M_1^5, \tag{6}$$

where K_j and M_j represents the j-th-order cumulant and j-th-order moment of the distribution, respectively.

Now, according to the definition of moments, a moment of $(i+j)$ th-order,

$$M_{i+j} = \langle (I(t))^{i+j} \rangle = c^{i+j}(0), \tag{7}$$

where 0 in the parenthesis of the autocorrelation function represents zero time delay. Because of the relation (7), one can say, using the relations (2) to (6), that a cumulant of a particular order represents a true autocorrelation function of that particular order at zero time delay. Then one can easily generalize the relations (2) to (6) for any time delay τ such that each of the resulting relations represents uniquely the true autocorrelation function of that particular order for any τ, and which reduces to cumulant of that particular order as $\tau \to 0$. One can call these autocorrelation functions as the "cumulant-like autocorrelation function" of that particular order. Then the definition of the cumulant-like autocorrelation function would be as given below (up to the 5th-order) obtained from the relations (2) to (6) :

Second order cumulant-like autocorrelation function

$$KC^{11}(\tau) = c^{11}(\tau) - K_1^2. \tag{8}$$

Third order cumulant-like autocorrelation function

$$KC^{ij}(\tau) = c^{ij}(\tau) - 3c^{11}(\tau)K_1 + 2K_1^3. \tag{9}$$

where $i+j=3$;

Fourth order cumulant-like autocorrelation function

$$KC^{ij}(\tau) = c^{ij}(\tau) - 4c^{pq}(\tau)K_1 - 3(c^{11}(\tau))^2 + 12c^{11}(\tau)K_1^2 - 6K_1^2 \tag{10}$$

where $i+j=4$, $p+q=3$; [for $i>=j$, $p+1=i$; while for $i<j$, $q+1=j$]

Fifth order cumulant-like autocorrelation function

$$KC^{ij}(\tau) = c^{ij}(\tau) - 5c^{pq}(\tau)K_1 - 10c^{mn}(\tau)c^{11}(\tau) + 20c^{mn}(\tau)K_1^2$$
$$+ 30(c^{11}(\tau))^2K_1 - 60c^{11}(\tau)K_1^3 + 24K_1^5, \tag{11}$$

where $i+J=5$, $p+q=4$, $m+n=3$, [for $i>j$, $p+1=i$, $m+1=p$; while for $i<j$, $q+1=j$, $n+1=q$].

In the above relations (8) to (11), $KC^{ij}(\tau)$ represents the cumulant-like autocorrelation function of order $(i+j)$ where, clearly, $KC^{ij}(0) = K_{i+j}$ (the cumulant of order $(i+j)$). Clearly the different formulae (8) to (11) remove the contribution of lower order autocorrelation functions in order to obtain cumulant-like autocorrelation function of that particular order. As an example, stated explicitly, one of the cumulant-like autocorrelation functions of the 5th-order, $KC^{41}(\tau)$, would be obtained from the relation (11) as,

$$KC^{41}(\tau) = c^{41}(\tau) - 5c^{31}(\tau)K_1 - 10c^{21}(\tau)c^{11}(\tau) + 20c^{21}(\tau)K_1^2$$
$$+ 30(c^{11}(\tau))^2K_1 - 60c^{11}(\tau)K_1^3 + 24K_1^5, \tag{12}$$

3. Discussion

As in the above mentioned cumulant-like autocorrelation functions, lower order autocorrelation functions has been subtracted out, they must possess better sensitivity to reveal true higher order effects of only that particular order. I encourage experimentalists to use these expressions wishing to obtain useful informations of higher order in the experimental signals by computing higher order autocorrelation functions. Such computations can be easily done for a signal from digitized records of its time series.

It is clear that each of the above expressions for the cumulant-like autocorrelation functions needs knowledge of K_1 (which is the first moment or the average value of the signal). Therefore, in a.c. coupled measurements, where the absolute average value of the signal cannot be measured accurately, these relations will not be useful. However, in that case one can use the relations between cumulants and central moments as given by Kendall and Stuart (1977) as an alternate basis for the definition of cumulant-like autocorrelation functions. Such autocorrelation functions have been defined and such intensity autocorrelation functions have been computed (Das and Abraham 1991) up to the 5th-order from the digitized intensity time series of the emission from a heavily saturated amplified spontaneous emission coming out from a 3 meter long mirrorless laser. From the computation, it has become clear that some memory and higher order effects are clearer in some of those intensity autocorrelation functions than in the intensity autocorrelation function $c^{ij}(\tau)$.

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